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A Random Covering Interpretation for the Phase Transition of the Random Energy Model

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The random energy model is related to a random covering of the real line. The phase transition is interpreted as the passage from a regime where a family of random intervals covers the line (high temperature) to a noncovering regime (low temperature).

KEY WORDS: Random energy model; random covering; Poisson point process.

The random energy model (REM) is introduced by Derrida[3,4] as a simple and solvable statistical model sharing some of the properties expected in spin glasses. The REM describes a system whose energy levels E_i are idependent and identically distributed random variables (with a Gaussian distribution in the original formulation) and the partition function is expressed as a statistical sum over 2^N levels,

$$Z(\beta) = \sum_{i=1}^{2^N} \exp(-\beta E_i)$$

Recently, Ruelle[11] reformulated the REM in terms of Poisson distributions. He rigorously showed that if the random variables E_i have a Poisson distribution on the real line with density $\varphi(x) = \rho \exp(\rho x)$, one can parametrize the probability structure of the model through the unique positive parameter given by $\alpha = \rho/\beta$. In a recent work, Galves *et al.*[6] found, among other things, a scaling allowing one to prove that the Boltzmann factors $l_i = \exp(-\beta E_i)$ in the $N \to \infty$ limit are realizations of a

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Poisson point process and interpreted the parameter ρ as the inverse critical temperature β_c.

In this note, the Boltzmann factors contributing to the statistical sum, are considered as the lengths of random intervals on $\mathbf{R}$. These random intervals are placed on the real line following a poisson process. The distribution of their lengths is governed by a positive measure over $\mathbf{R}^+$ having a density $\varphi(y)$ w.r.t. the Lebesgue measure given by $\varphi(y) = \gamma\alpha/y^{1+\alpha}$ $(0 < \gamma \leqslant 1)$, and the distribution of their positions is governed by the Lebesgue measure.

It is shown in the following that, for $\alpha > 1$, the real line is almost surely covered by the random intervals and the Hausdorff dimension of the set of uncovered points is zero. In contrast, for $\alpha \leqslant 1$ the set of uncovered points has Hausdorff dimension one. This provides a clear geometrical picture for the phase transition of the REM: in fact, it establishes that 1 is a critical value for the parameter α governing the distribution of the random lengths. Now, as α is given by ρ/β, the previous result is in agreement with the results of ref. 6, where ρ is identified with the critical inverse temperature.

In the following, the REM will be associated with a Poisson point process defined on the upper half-plane $\mathbf{R} \times \mathbf{R}^+$.

Remark first that the Boltzmann factors $l_i = \exp(-\beta E_i)$ are nonnegative; it is therefore natural to associate them with random variables distributed according to a measure supported by $\mathbf{R}^+$. As such, in accordance with ref. 11, choose the measure given by

$$\mu(l \in dy) = \gamma\alpha y^{-(\alpha+1)}\, dy, \qquad 0 < \gamma \leqslant 1$$

Now, the l_i, being nonnegative, can be interpreted as the lengths of random intervals. However, intervals are defined not only by their lengths, but by their origins (or their centers) as well. To get a clear geometrical insight of the phase transition, consider an infinite sequence of random variables x_i that will be interpreted as the origins of the intervals. Moreover, the choice of the probability measure of x_i is not relevant; what is merely required is to be a Radon measure on $\mathbf{R}$. The simplest measure available is the Lebesgue measure λ over $(\mathbf{R}, \mathscr{B}(\mathbf{R}))$. Hence, the REM that will be used in the sequel is given by the following.

Definition 1. Let λ be the Lebesgue measure on $(\mathbf{R}, \mathscr{B}(\mathbf{R}))$ and μ a measure on $(\mathbf{R}^+, B(\mathbf{R}^+))$ given by $\mu(l \in dy) = \gamma\alpha y^{-(1+\alpha)}\, dy$, with $0 < \gamma \leqslant 1$ and $\alpha \in \mathbf{R}^+$. Denote by ν the measure $\lambda \otimes \mu$ on $(\mathbf{R} \times \mathbf{R}^+, \mathscr{B}(\mathbf{R} \times \mathbf{R}^+))$. The random energy model is the Poisson point process $(x_i, l_i)_{i \in \mathbf{N}}$ on $\mathbf{R} \times \mathbf{R}^+$ of intensity $\nu = \lambda \otimes \mu$.

One can now consider the open random intervals $]x_i, x_i + l_i[$ where the origins and lengths (x_i, l_i) are obtained by means of the Poisson point

process of the REM. We call these intervals covering intervals and if C denotes their union, one can ask whether $C = \mathbf{R}$ or $C \neq \mathbf{R}$ almost surely.

The problem of the a.s. covering of the line by random intervals associated a Poisson point process was first posed and studied by Mandelbrot.[10] For locally bounded measures on $\mathbf{R}^+$, Shepp[2] gives a necessary and sufficient condition for a.s. covering. For the measure μ, this condition will be recovered at high temperature.

In the physics literature, the covering of the real line by Poisson distributed intervals is studied in the one-dimensional percolation models by Aizenman and Newman.[1]

Recently, Kahane[7] studied the covering problem in relation to the decomposition of positive measures on $\mathbf{R}$ into regular and singular parts. Here, this decomposition will be related to the low (resp. high) temperature behavior of the REM. This situation arises when a positive martingale given as product of independent weight functions acts on a positive Radon measure. One can define such a *weight* function of mean one for every point $x \in \mathbf{R}$ as follows.

Suppose that the random variables l_i are distributed according to the regularized measure

$$(\mathbb{1}_{[\varepsilon,\infty)}\mu)(l \in dx) = \gamma\alpha \mathbb{1}_{[\varepsilon,\infty)}(x) x^{-(1+\alpha)}\, dx$$

The union of the covering intervals $C_\varepsilon = \bigcup]x_i, x_i + l_i[$ is then indexed by ε and the weight function is given by

$$G_\varepsilon(x) = \mathbb{1}_{C_\varepsilon^c}(x)/P(x \notin C_\varepsilon)$$

It is easy to see that the above functions define a positive martingale of mean one. For a positive Radon measure τ on $\mathbf{R}$, one can consider the process $G_\varepsilon(x)\, \tau(dx)$ indexed by ε and let $G = \lim_{\varepsilon \to 0} G_\varepsilon \tau$, which exists by standard martingale convergence arguments. In the case where $G = 0$ a.s. (i.e., G is degenerate), the measure τ is called G_ε-singular. An interesting situation arises when $EG\tau = \tau$ (i.e., the martingale is uniform integrable); in that case, the operator mapping $\tau \mapsto EG\tau$ is a projection on the space of positive Radon measures of $\mathbf{R}$. We then say that the operator G is fully acting on τ (or that it is G_ε-regular). This can be summarized in the following theorem.

Theorem 1.[8] Given the process G_ε and a positive Radon measure τ on $(\mathbf{R}, \mathscr{B}(\mathbf{R}))$, there is a unique decomposition of G_ε into a sum of two positive martingales

$$G_\varepsilon = G_\varepsilon^r + G_\varepsilon^s$$

such that the corresponding operators $G^r = \lim_{\varepsilon \to 0} G^r_\varepsilon$ and $G^s = \lim_{\varepsilon \to 0} G^s_\varepsilon$ are respectively fully acting and degenerate on τ.

The above setting is a particular simple application of general formalism used for the study of multiplicative chaos.[2,8] The full action (resp., degeneracy) of the operator G_ε, as a function of the parameter α, corresponds to the noncovering (resp., complete covering) of the line by $C = \lim_{\varepsilon \to 0} C_\varepsilon$.

One can now separate the intervals l_i into short (resp., large) ones in accordance with the manner in which the energies E_i take positive (resp., negative) values. This separation will be relevant only in the low-temperature region ($a < 1$), where the real line is not covered by short intervals. Using this remark, we write for $\Lambda < \infty$ the measure μ as the sum $\mu_L + \mu_H$, with

$$\mu_L(dy) = \gamma \mathbb{1}_{[0,\Lambda)}(y)\, y^{-(1+\alpha)}\, dy$$

$$\mu_H(dy) = \gamma \mathbb{1}_{[\Lambda,\infty)}(y)\, y^{-(1+\alpha)}\, dy$$

Correspondingly, the process G_ε is split into $G^L_\varepsilon + G^H_\varepsilon$, where the variables l_i are distributed according to the regularized measures $\mathbb{1}_{[\varepsilon,\infty)}\mu_L$ and $\mathbb{1}_{[\varepsilon,\infty)}\mu_H$. The set C_ε is also split into $C^H_\varepsilon \cup C^L_\varepsilon$. Similarly, we define the uncovered set $U^L = \lim_{\varepsilon \to 0}(\mathbf{R} \backslash C^L_\varepsilon)$. We have the following result.

Proposition 1. With probability one, the Hausdorff dimension d_H of the uncovered set U^L is

$$d_H = \begin{cases} 0 & \text{for } \alpha > 1 \\ 1 - \gamma & \text{for } \alpha = 1 \\ 1 & \text{for } \alpha < 1 \end{cases}$$

Proof. Using Corollary 3 of ref. 5, we have for the Hausdorff dimension d_H of the uncovered set

$$d_H = 0 \vee \sup \left\{ \delta : v^{1-\delta} \exp\left(\int_v^\infty \mu_L(y, \infty]\, dy \right) \to 0 \text{ as } v \to 0 \right\}$$

The proposition follows immediately by substituting the explicit form of the measure μ_L into the above formula. ∎

The interpretation of phase transition as a random covering problem is summarized in the following theorem.

Theorem 2. Almost surely:

(i) For $\alpha > 1$, $\lim_{\varepsilon \to 0} C^L_\varepsilon = \mathbf{R}$ and the Lebesgue measure on $\mathbf{R}$ is G^L_ε-singular.

(ii) For $\alpha \leqslant 1$, $\lim_{\varepsilon \to 0} C_\varepsilon^{\mathrm{L}} \neq \mathbf{R}$ and the Lebesgue measure is $G_\varepsilon^{\mathrm{L}}$-regular.

An intermediate technical result is given in the following lemma.

Lemma 1. For $\alpha \leqslant 1$, $\int_0^1 G_\varepsilon^{\mathrm{L}}(x)\, dx \in L^2(\Omega)$.

Proof. We have

$$E\left[\int_0^1 G_\varepsilon^{\mathrm{L}}(x)\, dx\right]^2 = \int_0^1 dx \int_0^1 dy \frac{E(\mathbb{1}_{C_\varepsilon^{\mathrm{L}}}(x))\, E(\mathbb{1}_{C_\varepsilon^{\mathrm{L}}}(y))}{P(x \in C_\varepsilon^{\mathrm{L}})\, P(y \in C_\varepsilon^{\mathrm{L}})}$$

Using translation invariance and the definition of the Poisson point process, we find for the right-hand side

$$\frac{2}{P^2(0 \in C_\varepsilon^{\mathrm{L}})} \int_0^1 dx\, (1-x)\, P(0 \in C_\varepsilon^{\mathrm{L}}; x \in C_\varepsilon^{\mathrm{L}})$$

$$\leqslant 2 \int_0^1 \exp\left(\int_{x \vee \varepsilon}^{A} 1_{[\varepsilon, \infty)}(y)\, \mu_{\mathrm{L}}([y, A])\, dy\right)$$

Using now the explicit expression for $\mu_{\mathrm{L}}([y, A])$, it is easy to see that the above quantity is finite for $\alpha \leqslant 1$. ∎

Proof of Theorem 2. (i) For $\alpha > 1$ one has

$$\int_0^1 dx \exp\left(\int_{\varepsilon \vee x}^{\infty} \mathbb{1}_{[\varepsilon, \infty)} \mu_{\mathrm{L}}(y, \infty)\, dy\right) = \infty$$

Therefore, the theorem of ref. 12 can be used to show that $\lim_{\varepsilon \to 0} C_\varepsilon^{\mathrm{L}} = \mathbf{R}$ with probability one. Consequently, the martingale $(G_\varepsilon^{\mathrm{L}} \tau)([0, 1])$ fails to be square integrable and the singularity of the Lebesgue measure for $\alpha > 1$ follows immediately.

(ii) From the previous lemma and Doob's inequality for square-integrable martingales, the $G_\varepsilon^{\mathrm{L}}$-regularity of the Lebesgue measure follows. ∎

Now, if we consider the process $G_\varepsilon^{\mathrm{H}}$, we have the following result.

Proposition 2. For every α, $\lim_{\varepsilon \to 0} C_\varepsilon^{\mathrm{H}} = \mathbf{R}$ almost surely.

The phase transition of the REM is related to the random covering if one considers only short random intervals. The use of a cutoff in the lengths of the covering intervals can be understood as a kind of stability condition, since it corresponds to a bound from below the random energy levels. In the absence of any cutoff, some energy levels can be arbitrarily

negative. Hence, the system freezes into these configurations. This remark explains why the model does not exhibit a phase transition (in the sense defined in the previous section) in this case. At the critical point ($\alpha = 1$), and only there, the Hausdorff dimension of the set of uncovered points depends on the parameter γ. The use of this parameter can be considered as a global fine tuning of the mean length of the covering intervals. It is then natural that the parameter γ is relevant only at the critical point.

The previous setting can be applied for the study of other random models, such as the generalized random energy model (GREM), percolation problems, etc.[9]

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